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Similarity of Quadratic Forms and Half-Neighbors

Detlev W. Hoffmann*

Equipe de Mathématiques de Besançon, UMR 6623 du CNRS,
Université de Franche-Comté, 16, route de Gray,

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1. INTRODUCTION

Let F be a field of characteristic $\neq 2$ and let I^nF denote the nth power of the ideal IF of even-dimensional forms in the Witt ring WF of F. The Arason–Pfister Hauptsatz [AP] states that every quadratic form over F of dimension $< 2^n$ which lies in I^nF is hyperbolic. As a consequence, one has that if φ and ψ are forms over F with $\dim \varphi = \dim \psi < 2^n$ and if there exists an $a \in F^*$ such that $\varphi \equiv a\psi \pmod{I^{n+1}F}$, then $\varphi \simeq a\psi$, i.e., φ and ψ are similar. The question arises whether this conclusion still holds for $\dim \varphi = \dim \psi = 2^n$. In this case, if $\varphi \equiv a\psi \pmod{I^{n+1}F}$ then it follows from [AP, Kor. 3] that $\varphi \perp -a\psi$ is similar to an $(n+1)$-fold Pfister form. We will then say that φ and ψ are half-neighbors (of each other) and we will write $\varphi \underset{hn}{\sim} \psi$ (cf. Definition 2.1). It is not hard to show that if $n \leq 2$ then two half-neighbors are similar. To our knowledge, the first counterexamples for $n \geq 3$ were given by Izhboldin [I]. Shortly after, counterexamples of a rather simple nature were found by the author and it turned out that constructing such examples is in fact not too difficult. So the question arose under which circumstances half-neighbors are similar. In other words, suppose φ and ψ are half-neighbors, and, say, φ has a certain property. Does this property imply that φ is similar to ψ, or are there counterexamples with this particular property? We also ask the question whether certain properties of F pertaining to quadratic forms imply that half-neighbors over F are always similar.

* E-mail: detlev@math.univ-fcomte.fr.

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The interest in these questions was initially triggered by recent results of Laghribi. Let φ and ψ be anisotropic 8-dimensional forms over F with $\varphi \in I^2F$. Then it was shown in [L1, L2] that φ becomes isotropic over $F(\psi)$, the function field of ψ over F, if and only if ψ is similar to a 3-fold Pfister form π and φ contains a Pfister neighbor of π, or φ and ψ are half-neighbors. It is well known that the index of the Clifford algebra of φ as above is 1 (in which case φ itself is similar to a 3-fold Pfister form), 2, 4, or 8. If the index is 1, then it is not difficult to see that the above result just says that φ and ψ are similar to the same 3-fold Pfister form. In particular, φ and ψ being half-neighbors implies that φ and ψ are similar, a statement which is still true if the index is 2 (cf. [L1, Corollary 1] or [H5, Theorem 3.3]). But it turns out that there are counterexamples to similarity in the cases of index 4 and 8 (cf. Section 4). It should be noted that these counterexamples cannot be obtained by Izhboldin's method [I], and that their construction requires some deep results from the theory of division algebras [T, M, Ka].

In Section 2, we will give some characterizations and prove some basic properties of half-neighbors. Sections 3 and 4 deal with general methods of constructing nonsimilar half-neighbors with certain given properties. For example, we will show that to each $n \geq 3$ and each m, $1 \leq m < n$, there exist nonsimilar half-neighbors of dimension 2^n and degree m over suitable fields. We will also construct examples of nonsimilar half-neighbors in I^2 whose Clifford algebras have a prescribed index. The constructions of these examples, though somewhat technical in nature, are of interest in their own right since they employ a variety of techniques and results drawing from the theory of quadratic forms, division algebras, and function fields. For instance, Springer's theorem on quadratic forms over fields with a nondyadic 2-henselian valuation figures prominently. Some examples depend on the existence of triquadratic field extensions M/F for which the Brauer group complex has a certain nontrivial homology group $N_2(M/F)$ (cf. [ELTW]). We will also make use of Merkurjev's index reduction results for division algebras over function fields of quadrics.

In Section 5, we show how certain properties of F imply that certain types of half-neighbors are always similar. We will show that if the Hasse number $\tilde{u}$ of F (i.e., the maximum dimension of anisotropic totally indefinite quadratic forms over F) is ≤ 6, then half-neighbors are always similar. Thus, for example, half-neighbors over global fields or over fields of transcendence degree ≤ 1 over the reals are always similar.

Finally, in Section 6, we will compile mostly known results regarding certain equivalence relations which one can define on the set of quadratic forms of a given dimension. One such equivalence relation is defined for forms of dimension 2^n by "$\underset{hn}{\sim}$" (cf. Proposition 2.3). Others are as follows: similarity of quadratic forms where we write $\varphi \underset{sim}{\sim} \psi$ if φ is similar to ψ;

the equivalence relation $\underset{bir}{\sim}$ derived from birational equivalence of the projective quadrics defined by the quadratic forms (in other words, $\varphi \underset{bir}{\sim} \psi$ if $F(\varphi) \cong F(\psi)$); and the equivalence relation $\underset{stb}{\sim}$, where we have $\varphi \underset{stb}{\sim} \psi$ if the associated quadrics are stably birationally equivalent (one has $\varphi \underset{stb}{\sim} \psi$ iff both $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic, cf. [O, Sect. 3]). One of the important questions in the theory of quadratic forms and their function fields is whether stably birational equivalent forms of the same dimension are already birationally equivalent (J. Ohm [O] calls this the "quadratic Zariski problem"). It is this question which serves as a further motivation for the study of half-neighbors and their (non)similarity: on the one hand, half-neighbors are easily seen to be stably birationally equivalent (Corollary 2.6); on the other hand, similar forms are birationally equivalent. In some sense, half-neighbors are quite close to being similar (see the introductory remarks), so it seems natural to ask the question of birational equivalence for them. If half-neighbors had always been similar, this task would have been trivial. The existence of nonsimilar half-neighbors shows that there is still work to be done. In fact, it should be noted that at this point no counterexamples to the quadratic Zariski problem are known at all.

Our notations will generally follow those introduced in Lam's book [Lam] and Scharlau's book [S], and we assume the reader to be familiar with the basic results from the algebraic theory of quadratic forms which can be found there, such as the theory of Pfister forms, the Cassels–Pfister subform theorem, the Arason–Pfister Hauptsatz, and function fields of quadratic forms.

By the usual abuse of notation, we will use the same symbol for a quadratic form and for its class in the Witt ring. $\varphi \simeq \psi$, $\varphi \perp \psi$, and $\varphi \subset \psi$ denote, respectively, isometry of forms, their orthogonal sum, and the fact that φ is a subform of ψ, i.e., there exists a form τ such that $\psi \simeq \tau \perp \varphi$. $i_W(\varphi)$, φ_{an}, $d_{\pm}\varphi$, and $c(\varphi)$ denote the Witt index, the anisotropic part, the signed discriminant, and the Clifford invariant of φ, respectively. By $\operatorname{ind} c(\varphi)$ we mean the index of the division algebra whose class in the Brauer group $\operatorname{Br} F$ equals $c(\varphi)$.

If φ is a form over F and K/F is a field extension, then φ_K is the form over K obtained from φ by scalar extension, and $W(K/F)$ denotes the kernel of the ring homomorphism $WF \to WK$ induced by scalar extension.

If $\dim \varphi \geq 2$ and $\varphi \not\simeq \langle 1, -1\rangle$, then the function field $F(\varphi)$ of φ is the function field of the projective quadric defined to be the equation $\varphi = 0$, and we put $F(\varphi) = F$ if $\dim \varphi \leq 1$ or $\varphi \simeq \langle 1, -1\rangle$.

We will use the convention $\langle\langle a_1, \dots, a_n\rangle\rangle$ to denote the n-fold Pfister form $\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$. The set of forms isometric (resp. similar) to n-fold Pfister forms over F will be denoted by P_nF (resp. GP_nF).

If φ is a form over F, then $D_F(\varphi) = \{a \in F^* | \langle a \rangle \subset \varphi\}$ is the set of nonzero elements represented by φ, and $G_F(\varphi) = \{a \in F^* | \varphi \simeq a\varphi\}$ is its group of similarity factors. If there is no confusion with regard to the field F, then we tend to omit the subscript.

Some familiarity with generic splitting of quadratic forms is assumed. The basic references are [Kn1; Kn2; S, Sects. 4.6, 4.7]. In particular, we will need or refer to generic splitting towers, Knebusch's ideals $J_n F$, and the degree function deg defined via the filtration on WF by these ideals. The corresponding degree function which arises from the filtration given by the $I^n F$'s will be denoted by deg$'$.

If F is formally real, then X_F denotes the space of all orderings of F and $\operatorname{sgn}_P \varphi$ is the signature of a form φ over F at the ordering $P \in X_F$. In Section 5, we will need some basic facts about fields with finite Hasse number $\tilde{u}$ and SAP fields, as can be found, for example, in [ELP; E, Sect. 4].

2. BASIC PROPERTIES OF HALF-NEIGHBORS

DEFINITION 2.1. Let φ and ψ be forms over F of dimension 2^n. φ and ψ are called *half-neighbors* (*of each other*), in symbols $\varphi \underset{hn}{\sim} \psi$, if there exists $a \in F^*$ such that $\varphi \perp -a\psi \in GP_{n+1}F$.

Remark 2.2. Let φ and ψ be forms over F of dimension 2^n.

(i) Knebusch [Kn2, Definition 8.7] calls φ and ψ conjugate if $\varphi \perp -\psi \simeq a\rho$ for some $a \in F^*$ and some $\rho \in P_{n+1}F$, in which case he calls φ and ψ half-neighbors of ρ. Our definition of two forms being half-neighbors is a little less restrictive than Knebusch's definition of two forms being conjugate. Note that in our definition half-neighbors always come in pairs. Fitzgerald [F1, F2] calls a form of dimension 2^n a conjugate neighbor if there exists an *anisotropic* form $\rho \in GP_{n+1}F$ such that $\varphi \subset \rho$.

(ii) If φ and ψ are similar, say $\varphi \simeq a\psi$ for some $a \in F^*$, then $\varphi \underset{hn}{\sim} \psi$. In fact, $\varphi \perp -a\psi$ is isometric to the hyperbolic $(n+1)$-fold Pfister form.

(iii) If $\varphi \underset{hn}{\sim} \psi$ and, say, φ is isotropic, then $\varphi \underset{sim}{\sim} \psi$. In fact, let $a \in F^*$ such that $\varphi \perp -a\psi \simeq \pi \in GP_{n+1}F$. Since φ is isotropic, π is also isotropic and hence hyperbolic. Thus we clearly have $\varphi \simeq a\psi$.

(iv) If $n = 0$ or 1 then $\varphi \underset{hn}{\sim} \psi$ iff $\varphi \underset{sim}{\sim} \psi$. This is trivial for $n = 0$ and an easy determinant argument for $n = 1$.

PROPOSITION 2.3. $\underset{hn}{\sim}$ *defines an equivalence relation on all forms over F of dimension 2^n.*

Proof. Clearly, $\underset{hn}{\sim}$ is symmetric, and it is reflexive because if φ is a form of dimension 2^n then $\varphi \perp -\varphi$ is isometric to the hyperbolic $(n+1)$-fold Pfister form. Now let φ, ψ, τ be forms over F of dimension 2^n with $\varphi \underset{hn}{\sim} \psi$ and $\psi \underset{hn}{\sim} \tau$. Say, $\varphi \perp -a\psi \simeq \rho \in GP_{n+1}F$ and $\psi \perp -b\tau \simeq \sigma \in GP_{n+1}F$ for some $a, b \in F^*$. Using Witt cancellation, we then get in WF that $\varphi \perp -ab\tau = \rho \perp a\sigma \in I^{n+1}F$. Now $\dim(\varphi \perp -ab\tau) = 2^{n+1}$ and by the Arason–Pfister Hauptsatz we have $\varphi \perp -ab\tau \in GP_{n+1}F$. Hence, $\varphi \underset{hn}{\sim} \tau$, which proves transitivity. ∎

PROPOSITION 2.4 (compare [Kn2, Theorem 8.9]). *Let φ and ψ be forms over F with* $\dim \varphi = \dim \psi = m$. *Then the following are equivalent*:

(i) $\varphi \underset{sim}{\sim} \psi$ *or* $\varphi \underset{hn}{\sim} \psi$;

(ii) $\varphi \perp -a\psi \in W(F(\varphi)/F) \cap W(F(\psi)/F)$ *for some* $a \in F^*$.

Furthermore, if φ and ψ are anisotropic, then the above statements are equivalent to

(iii) $\varphi \perp -a\psi \in W(F(\varphi)/F) \cup W(F(\psi)/F)$ *for some* $a \in F^*$.

Proof. Obviously, (ii) ⇒ (iii) holds even if one of the forms is isotropic. Also, the above is trivially true if $m = 1$. So let us assume throughout that $m \geq 2$.

(i) ⇒ (ii). If $\varphi \underset{sim}{\sim} \psi$ this is trivially true as there exists $a \in F^*$ such that $\varphi \perp -a\psi$ is hyperbolic. So suppose that $\varphi \underset{hn}{\sim} \psi$. By Remark 2.2(iii) and (iv), we may assume that both φ and ψ are anisotropic and that $m = 2^n \geq 4$. Let $a \in F^*$ and $\pi \in GP_{n+1}F$ such that $\varphi \perp -a\psi \simeq \pi$. Since $\varphi_{F(\varphi)}$ is isotropic, we have that $\pi_{F(\varphi)}$ is isotropic and hence hyperbolic. In particular, $\varphi \perp -a\psi \in W(F(\varphi)/F)$. By symmetry, $\varphi \perp -a\psi \in W(F(\psi)/F)$ as well.

(ii) resp. (iii) ⇒ (i). If one of the forms is isotropic, then $W(F(\varphi)/F) \cap W(F(\psi)/F) = 0$ and (ii) readily implies $\varphi \underset{sim}{\sim} \psi$. So we may assume that both forms are anisotropic in which case it suffices to show that (iii) implies (i). Say, $\rho \simeq (\varphi \perp -a\psi)_{an} \in W(F(\varphi)/F)$. We may furthermore assume that φ is not similar to ψ so that ρ is not hyperbolic. It follows from [F1, Theorem 1.6] that $\rho \in GP_{n+1}F$ for some $n \geq 1$. Since $2m \geq \dim \rho = 2^{n+1}$, we have $m \geq 2^n$. If $m = 2^n$ then $\rho \simeq \varphi \perp -a\psi$ and hence $\varphi \underset{hn}{\sim} \psi$. So suppose $m > 2^n$. Note that $a\psi = \varphi \perp -\rho \in WF$ and by comparing dimensions, we see that $\varphi \perp -\rho$ is isotropic and thus $D(\rho) \cap D(\varphi) \neq \emptyset$. Since $\rho_{F(\varphi)}$ is hyperbolic, the Cassels–Pfister subform theorem then implies that $\rho \simeq \varphi \perp \chi$ for some form χ over F. Hence, $a\psi = -\chi \in WF$. But $\dim \psi = m > 2^n > 2^{n+1} - m = \dim \chi$. This yields that ψ is isotropic, a contradiction. ∎

Remark 2.5. (i) The implication (iii) ⇒ (i) in the above proposition generally does not hold any longer if we replace $W(F(\varphi)/F) \cup W(F(\psi)/F)$ by the ideal generated by this set, $W(F(\varphi)/F) + W(F(\psi)/F)$. In fact, let F be any field with at least 4 square classes, let $a, b \in F^*$ represent different nontrivial square classes. Then $\varphi \simeq \langle 1, -a \rangle$ and $\psi \simeq \langle 1, -b \rangle$ are anisotropic, but they are clearly not half-neighbors of each other (and thus also not similar), but obviously $\varphi \perp \psi \in W(F(\varphi)/F) + W(F(\psi)/F)$.

(ii) The implication (iii) ⇒ (i) generally also fails if one drops the assumption on φ and ψ being anisotropic. If φ is any anisotropic n-fold Pfister form, $n \geq 1$ over a suitable field F, and if ψ is the hyperbolic n-fold Pfister form over F, then φ and ψ are clearly not half-neighbors (and thus not similar), but $\varphi \perp \psi \in W(F(\varphi)/F) \subset W(F(\varphi)/F) \cup W(F(\psi)/F)$.

(iii) The implication (i) ⇒ (ii) generally fails if in statement (ii) one replaces $W(F(\varphi)/F) \cap W(F(\psi)/F)$ by the ideal $W(F(\varphi)/F) \cdot W(F(\psi)/F)$. In fact, using Corollary 3.6 below, one can construct a non-formally real field F with $u(F) = 16$ and nonsimilar half-neighbors φ and ψ of dimension 8 such that there are $\sigma, \tau \in P_3 F$ such that φ and ψ contain a Pfister neighbor of σ and τ, respectively. One immediately gets that $W(F(\varphi)/F) \subset W(F(\sigma)/F) = \sigma WF$ and $W(F(\psi)/F) \subset W(F(\tau)/F) = \tau WF$. In particular, $W(F(\varphi)/F) \cdot W(F(\psi)/F) \subset \sigma \otimes \tau WF$ with $\sigma \otimes \tau \in P_6 F$. Now $u(F) = 16 < 2^6 = \dim \sigma \otimes \tau$. Hence, $\sigma \otimes \tau$ is hyperbolic and thus $W(F(\varphi)/F) \cdot W(F(\psi)/F) = 0$. But $\varphi \perp -a\psi \neq 0$ in WF for all $a \in F^*$ as φ and ψ are not similar.

COROLLARY 2.6. *Let φ and ψ be forms over F of dimension 2^n. Then* $\varphi \underset{sim}{\sim} \psi \Rightarrow \varphi \underset{hn}{\sim} \psi \Rightarrow \varphi \underset{stb}{\sim} \psi$.

Proof. By Remark 2.2(ii), it suffices to show $\varphi \underset{hn}{\sim} \psi \Rightarrow \varphi \underset{stb}{\sim} \psi$. By Proposition 2.4, $\varphi \underset{hn}{\sim} \psi$ implies $\varphi_{F(\varphi)} \simeq a\psi_{F(\varphi)}$ and $\varphi_{F(\psi)} \simeq a\psi_{F(\psi)}$ for some $a \in F^*$. Since $\dim \varphi = \dim \psi$ and since $\varphi_{F(\varphi)}$ and $\psi_{F(\psi)}$ are isotropic, we thus have that $\psi_{F(\varphi)}$ and $\varphi_{F(\psi)}$ are isotropic as well, which implies $\varphi \underset{stb}{\sim} \psi$. ∎

PROPOSITION 2.7. *Let $n \geq 2$ and let φ and ψ be forms over F of dimension 2^n.*

(i) $\varphi \underset{hn}{\sim} \psi$ *iff there exist a form τ over F with* $\dim \tau \leq 2^{n+1} - 2n - 2$, $\pi \in P_n F$, *and* $a, u, v \in F^*$ *such that* $\varphi = \tau + u\pi$ *and* $a\psi = \tau + v\pi$.

(ii) *Let $1 \leq m \leq 2^{n-2}$ and suppose that φ contains a Pfister neighbor of dimension $2^{n-2} + m$. Then the statement in* (i) *holds with* $\dim \tau \leq 2^n + 2^{n-1} - 2m - 2$.

(iii) *Let $1 \leq m \leq 2^{n-1}$ and suppose that φ contains a Pfister neighbor of dimension $2^{n-1} + m$. Then the statement in* (i) *holds with* $\dim \tau \leq 2^n - 2m$.

Proof. In all cases, the "if" part follows from the fact that $\varphi \perp -a\psi = u\pi \perp -v\pi$ in WF. Indeed, $u\pi \perp -v\pi \in GP_{n+1}F$ and comparing dimensions yields $\varphi \perp -a\psi \simeq u\pi \perp -v\pi$. Thus, $\varphi \underset{hn}{\sim} \psi$. It remains to show the converse in the three cases. So let $\varphi \underset{hn}{\sim} \psi$ and let $a \in F^*$ and $\rho \in GP_{n+1}F$ such that $\varphi \perp -a\psi \simeq \rho$.

(i) Let us write $\varphi \simeq \chi \perp \eta$ with $\dim \chi = n + 1$, so that $\rho \simeq \chi \perp \sigma$ with $\dim \sigma = 2^{n+1} - n - 1$. Note that $\dim \sigma > 2^n$ as $n \geq 2$. Thus, σ is a Pfister neighbor of codimension $n + 1$ of the $(n + 1)$-fold Pfister form similar to ρ. By [AhO, Corollary 2.5], σ is a so-called special Pfister neighbor, i.e., there exist $\pi \in P_n F$, $v \in F^*$, and a form μ similar to a subform of π, such that $\sigma \simeq \mu \perp -v\pi$. Thus, $\rho \simeq \chi \perp \mu \perp -v\pi$ with $\dim(\chi \perp \mu) = 2^n$. It follows readily from the theory of Pfister forms that there exists $u \in F^*$ with $\chi \perp \mu \simeq u\pi$. We now put $\tau := \eta \perp -\mu$. One easily checks that $\dim \tau = 2^{n+1} - 2n - 2$ and that in WF we have $\varphi = \tau + u\pi$ and $a\psi = \tau + v\pi$.

(ii) Let χ be a Pfister neighbor of maximal dimension r contained in φ and write $\varphi \simeq \chi \perp \tilde{\eta}$. By assumption, $r \geq 2^{n-2} + m$. If $r \geq 2^{n-1} + 1$ we are in case (iii) below and we will obtain a form τ with $\dim \tau \leq 2^{n+1} - 2r \leq 2^n - 2 \leq 2^n + 2^{n-1} - 2m - 2$. If $r = 2^{n-1}$ then $\chi \in GP_{n-1}F$. Let $x \in D(\tilde{\eta})$. It follows that φ contains the subform $\chi \perp \langle x \rangle$ which is a Pfister neighbor of dimension $2^{n-1} + 1 > r$, a contradiction to the maximality of r. So we may assume that $2^{n-2} + m \leq r \leq 2^{n-1} - 1$. With x as above, the maximality of r implies that the subform $\chi \perp \langle x \rangle$ of φ is not a Pfister neighbor. Over $K = F(\chi \perp \langle x \rangle)$ we have that ρ_K is isotropic and hence hyperbolic, i.e., $\rho \in W(K/F)$. By [F2, Propositions 1.2 and 1.4(b)] and comparing dimensions, we can write $\rho \simeq \pi_1 \perp \pi_2$ with $\pi_i \in GP_n F \cap W(K/F)$, $i = 1, 2$. In fact, we may assume that $\chi \perp \langle x \rangle \subset \pi_1$ (cf. the proof of [F2, Proposition 1.2]). Let $u \in F^*$ and $\pi \in P_n F$ such that $u\pi \simeq$

π_1. Let now η and μ be forms over F of dimension $2^n - r - 1$ such that $\varphi \simeq \chi \perp \langle x \rangle \perp \eta$ and $u\pi \simeq \chi \perp \langle x \rangle \perp \mu$. Similar to above, we put $\tau := \eta \perp -\mu$ so that $\varphi = \tau + u\pi$, and we find $v \in F^*$ such that $a\psi = \tau + v\pi$. This time, however, we get $\dim \tau = 2(2^n - r - 1) \leq 2^n + 2^{n-1} - 2m - 2$.

(iii) Similarly as before, let us write $\varphi \simeq \chi \perp \eta$ with χ a Pfister neighbor of maximal dimension r contained in φ. By assumption, we have $r \geq 2^{n-1} + m$. Let $\pi \in P_nF$ such that χ is a Pfister neighbor of π and let $u \in F^*$ such that $u\pi \simeq \chi \perp \mu$ with a suitable form μ over F of dimension $2^n - r$. A similar reasoning as before yields that with $\tau \simeq \eta \perp -\mu$ we get $\varphi = \tau + u\pi$ and $a\psi = \tau + v\pi$ for a suitable $v \in F^*$. Furthermore, $\dim \tau = 2(2^n - r) \leq 2^n - 2m$. ∎

The study of half-neighbors is restricted to forms of dimension 2^n. The simplest examples of forms of that dimension are those similar to n-fold Pfister forms. Another class is that of so-called twisted Pfister forms which have been studied in [H6] and whose definition we will now recall. Let $1 \leq m < n$. φ is called a twisted (n,m)-Pfister form (or simply just (n,m)-Pfister form) if φ is anisotropic, $\dim \rho = 2^n$, and there exist anisotropic forms $\sigma \in P_nF$ and $\rho \in P_mF$ such that $\rho \simeq (\sigma \perp -\rho)_{an}$, in which case we say that φ is defined by (σ, ρ). The set of all forms isometric (resp. similar) to (n,m)-Pfister forms is denoted by $P_{n,m}F$ (resp. $GP_{n,m}F$). Note that $\varphi \in GP_{n,m}F$ iff φ is anisotropic of dimension 2^n and there exist anisotropic $\tilde{\sigma} \in GP_nF$ and $\tilde{\rho} \in GP_mF$ such that $\rho = \tilde{\sigma} + \tilde{\rho}$ in WF.

PROPOSITION 2.8. *Let $1 \leq m < n$, $\varphi \in GP_{n,m}F$, and ψ be a form over F of dimension 2^n. Then $\varphi \underset{hn}{\sim} \psi$ iff $\varphi \underset{sim}{\sim} \psi$.*

Proof. It suffices to prove the "only if" part. Let $\varphi = \sigma + \rho$ in WF with $\sigma \in GP_nF$ and $\rho \in GP_mF$. Comparing dimensions, we get $i_W(\sigma \perp \rho) = 2^{m-1}$. In particular, φ contains a Pfister neighbor of dimension $2^n - 2^{m-1} = 2^{n-1} + (2^{n-1} - 2^{m-1})$ contained in σ. By Proposition 2.7(iii), there exist a form τ of dimension $2^n - 2(2^{n-1} - 2^{m-1}) = 2^m$, $\pi \in P_nF$, and $u, v, w \in F^*$ such that, in WF, $\varphi = \tau + u\pi$ and $\psi = v\tau + w\pi$. We get $\varphi = \tau + u\pi = \sigma + \rho$, and since $\pi, \sigma, \rho \in I^mF$ we have $\tau \in I^mF$. But $\dim \tau = 2^m$ and thus $\tau \in GP_mF$. After scaling, we may assume $\tau \in P_mF$. Comparing dimensions shows that $\tau \perp u\pi$ is isotropic, i.e., there exists $x \in D(\tau) \cap D(-u\pi)$. Since τ and π are Pfister forms, we have $x \in G(\tau)$ and $-xu \in G(\pi)$. Hence, $\varphi = \tau + u\pi = x(\tau - \pi)$. Similarly, one finds $y \in F^*$ such that $\psi = y(\tau - \pi)$. This clearly yields $\varphi \underset{sim}{\sim} \psi$. ∎

COROLLARY 2.9. *Let φ and ψ be forms over F of dimension 2^n. Suppose φ contains a Pfister neighbor of codimension ≤ 1 of some n-fold Pfister form. Then $\varphi \underset{hn}{\sim} \psi$ iff $\varphi \underset{sim}{\sim} \psi$.*

Proof. It suffices to show the "only if" part. We may also assume that $n \geq 2$ and that φ and ψ are anisotropic. By Proposition 2.7(iii), we can write $\varphi = \tau + u\pi$ and $\psi = v\tau + w\pi$ in WF, with $\dim \tau = 2$, $\pi \in P_nF$ and $u, v, w \in F^*$. For dimension reasons, π must be anisotropic. If $\tau = 0$ then $\varphi \simeq u\pi \simeq uw\psi$. If τ is anisotropic then $\varphi \in GP_{n,1}F$ and $\varphi \underset{sim}{\sim} \psi$ by Proposition 2.8.

COROLLARY 2.10. *Let φ and ψ be forms over F of dimension 2^n, $n \geq 3$.*

(i) *If $n \leq 2$ then $\varphi \underset{hn}{\sim} \psi$ iff $\varphi \underset{sim}{\sim} \psi$.*

(ii) *If $n = 3$ then $\varphi \underset{hn}{\sim} \psi$ iff there exist $\pi \in P_3F$, a form τ over F with $\dim \tau \leq 8$, and $a, u, v \in F^*$ such that, in WF, $\varphi = \tau + u\pi$ and $a\psi = \tau + v\pi$.*

Proof. Any form of dimension ≤ 3 is a Pfister neighbor. Thus, (i) follows immediately from Corollary 2.9, and (ii) is a consequence of Proposition 2.7(ii). ∎

Let φ and ψ be half-neighbors of dimension 2^n. One of the main problems we are concerned with is to decide whether they are similar. By Proposition 2.7 and after scaling, we get a decomposition $\varphi = \tau + u\pi$ and $\psi = \tau + v\pi$ with $\pi \in P_nF$, a suitable form τ over F, and $u, v \in F^*$. The proposition below will give us a method of deciding whether φ and ψ are similar in terms of $G(\tau)$ and $G(\pi) = D(\pi)$, provided $\dim \tau < 2^n$. This knowledge will be useful later on in our constructions of nonsimilar half-neighbors.

LEMMA 2.11. *Let φ and ψ be forms of the same dimension over F and suppose there exist a form τ over F with* $\dim \tau < 2^n$, *$\pi \in I^nF$, and $u, v \in F^*$, such that $\varphi = \tau + u\pi$ and $\psi = \tau + v\pi$. Let $y \in F^*$. Then $\varphi \simeq y\psi$ iff $y \in G(\tau) \cap uvG(\pi)$. In particular, $\varphi \underset{sim}{\sim} \psi$ iff $G(\tau) \cap uvG(\pi) \neq \emptyset$, and $G(\varphi) = G(\psi) = G(\tau) \cap G(\pi)$.*

Proof. Suppose $\varphi \simeq y\psi$, $y \in F^*$. Then $\tau + u\pi = y\tau + yv\pi$ and hence $\tau - y\tau = yv\pi - u\pi = yv\pi \otimes \langle\langle -yuv\rangle\rangle \in I^{n+1}F$. Now $\dim(\tau \perp -y\tau) < 2^{n+1}$ and the Arason–Pfister Hauptsatz implies that $\tau \perp -y\tau$ is hyperbolic. We then have $\tau \simeq y\tau$ and $\pi \simeq yuv\pi$ which yields $y \in G(\tau)$ and $yuv \in G(\pi)$. Hence, $y \in G(\tau) \cap uvG(\pi)$. The converse is rather obvious, as are the remaining statements. ∎

3. CONSTRUCTION OF NONSIMILAR HALF-NEIGHBORS OF DIMENSION 2^n AND DEGREE $\leq n - 2$

In our constructions, we will make heavy use of the behavior of quadratic forms over fields of iterated power series using Springer's theorem [S, Chap. 6, Corollary 2.6]. Let E be any field of characteristic $\neq 2$ and let $F = E((x_1))\cdots((x_n))$ be the field of iterated power series in the variables $x_1, \dots, x_n$. Let $G = (\mathbf{Z}/2\mathbf{Z})^n$ with the fixed standard basis $\{e_1, \dots, e_n\}$ over $\mathbf{Z}/2\mathbf{Z}$. Each $g \in G$ can be written in a unique way as $g = \sum_{i=1}^n \lambda_i e_i$ with $\lambda_i \in \{0,1\}$. For any such g, we define the monomial $X_g = \prod_{i=1}^n x_i^{\lambda_i}$. In particular, $X_0 = 1$ and $X_{e_i} = x_i$. It is well known that, with this notation, for each $u \in F^*$ there exist $v \in E^*$ and $g \in G$ such that $u = vX_g \in F^*/(F^*)^2$. Furthermore, if $u' \in F^*$ and if $v' \in E^*$ and $g' \in G$ such that $u' = v'X_{g'} \in F^*/(F^*)^2$, then $uu' = vv'X_{g+g'} \in F^*/(F^*)^2$.

If φ is a quadratic form over F, then, by Springer's theorem, there exist forms φ_g over E, $g \in G$, such that $\varphi \simeq \perp_{g \in G} X_g \varphi_g$. We call the φ_g's the residue forms of φ. If $\varphi \simeq \perp_{g \in G} X_g \varphi'_g$ is another such decomposition, then $\varphi_g = \varphi'_g$ in WE for all $g \in G$. Furthermore, φ is anisotropic over F iff all the φ_g are anisotropic over E, in which case we even have $\varphi_g \simeq \varphi'_g$ over E for all $g \in G$. Generally, we get for the Witt index $i_W(\varphi)$ over F that $i_W(\varphi) = \sum_{g \in G} i_W(\varphi_G)$, where on the right hand side the Witt indexes are computed over E.

The following lemma is rather obvious and we omit its proof.

LEMMA 3.1. *With the same notations as above, let $\varphi \simeq \perp_{g \in G} X_g \varphi_g$ and $\psi \simeq \perp_{g \in G} X_g \psi_g$ be anisotropic forms over F. Then φ is similar to ψ iff there exist $v \in E^*$ and $h \in G$ such that $v\varphi_g \simeq \psi_{g+h}$ over E for all $g \in G$. In particular, if there exists $g' \in G$ such that $\dim \psi_{g'} \neq \dim \varphi_g$ for all $g \in G$, then φ and ψ are not similar.*

COROLLARY 3.2. *Let E be any field of characteristic $\neq 2$ and let F/E be any unirational field extension of E and φ and ψ be forms over E. Then $\varphi \underset{sim}{\sim} \psi$ over E iff $\varphi_F \underset{sim}{\sim} \psi_F$ over F.*

Proof. Since anisotropic forms stay anisotropic over rational and thus also over unirational field extensions, we may assume that φ and ψ are anisotropic. It clearly suffices to show the "if" part in the case $F = E(x)$, the rational function field in one variable x over E. Now if φ and ψ are similar over $E(x)$, then they are also similar over the bigger field $E((x))$. Since they are defined over E, it follows immediately from Lemma 3.1 that they are similar over E. ∎

Another straightforward consequence, again stated without proof, is the following.

COROLLARY 3.3. *Let φ and ψ be forms over a field E of characteristic $\neq 2$ with $\dim \varphi = \dim \psi$, let x be a variable and F be any field with $E(x) \subset F \subset E((x))$. Then $\varphi \underset{sim}{\sim} \psi$ over E iff $\varphi_F \otimes \langle\langle x \rangle\rangle \underset{sim}{\sim} \psi_F \otimes \langle\langle x \rangle\rangle$ over F.*

THEOREM 3.4. *Let n, m be integers with $1 \le m \le n-2$, and let E be a field which is either formally real, or non-formally real with $u(E) \ge 4$. If E is formally real, let F be any field with $E(x,y) \subset F \subset E((x))((y))$; if E is non-formally real, let F be any field with $E(x_1,\dots,x_{n-1}) \subset F \subset E((x_1))\cdots((x_{n-1}))$ (where x, y resp. $x_1,\dots,x_{n-1}$ are variables). Then there exist forms $\varphi_{n,m}$ and $\psi_{n,m}$ over F such that*

- $\dim \varphi_{n,m} = \dim \psi_{n,m} = 2^n$;
- *$\varphi_{n,m}$ and $\psi_{n,m}$ are nonsimilar half-neighbors of each other*;
- $\deg \varphi_{n,m} = \deg' \varphi_{n,m} = \deg \psi_{n,m} = \deg' \psi_{n,m} = m$;
- *$\varphi_{n,1}$ and $\psi_{n,1}$ contain Pfister neighbors of codimension 2 of some n-fold Pfister form.*

Proof. First, let E be formally real. For our construction, it will suffice to consider the case $F = E((x))((y))$. Let $\pi_n := \langle\langle 1,\dots,1,x\rangle\rangle \in P_nF$, $\tau := -\langle 1,1,y,xy\rangle$, and

$$\varphi_{n,1} := (\tau \perp \pi_n)_{an} \simeq \langle \underbrace{1,\dots,1}_{2^{n-1}-2}, \underbrace{x,\dots,x}_{2^{n-1}}, -y, -xy\rangle,$$

and

$$\psi_{n,1} := (\tau \perp y\pi_n)_{an} \simeq \langle -1, -1, \underbrace{y,\dots,y}_{2^{n-1}-1}, \underbrace{xy,\dots,xy}_{2^{n-1}-1}\rangle.$$

Using Witt cancellation and by comparing dimensions, we get $\varphi_{n,1} \perp -\psi_{n,1} \simeq \pi_n \perp -y\pi_n \in P_{n+1}F$, hence $\varphi_{n,1}$ and $\psi_{n,1}$ are half-neighbors of each other. By comparing dimensions of the residue forms, it follows that they are not similar (cf. Lemma 3.1). Furthermore, by construction, $\varphi_{n,1}$ and $\psi_{n,1}$ contain Pfister neighbors of codimension 2 of π. Also,

$$\varphi_{n,1} \equiv \psi_{n,1} \equiv \tau \equiv \langle 1, -x\rangle \qquad (\text{mod } I^2F) \text{ resp. } (\text{mod } J_2F).$$

Obviously, $\langle 1-x\rangle \simeq \langle\langle -x\rangle\rangle \in P_1F$ is anisotropic. Hence, $\deg \varphi_{n,1} = \deg \psi_{n,1} = \deg' \varphi_{n,1} = \deg' \psi_{n,1} = 1$.

Let now $m \ge 2$ and $\sigma_{m-1} := \langle\langle 1,\dots,1\rangle\rangle \in P_{m-1}F$. We now put $\varphi_{n,m} := \sigma_{m-1} \otimes \varphi_{n-m+1,1}$ and $\psi_{n,m} := \sigma_{m-1} \otimes \psi_{n-m+1,1}$. These forms have dimension 2^n and are anisotropic by Springer's theorem. They are half-neighbors as they are obtained by multiplying half-neighbors by the same Pfister form. Since the dimensions of the residue forms of $\varphi_{n,m}$ resp. $\psi_{n,m}$ are just the dimensions of the residue forms of $\varphi_{n-m+1,1}$ resp. $\psi_{n-m+1,1}$ multiplied by 2^{m-1}, Lemma 3.1 again readily implies that $\varphi_{n,m}$ and $\psi_{n,m}$ are not similar. Furthermore,

$$\varphi_{n,m} \equiv \psi_{n,m} \equiv \sigma_{m-1} \otimes \langle\langle -x\rangle\rangle \qquad (\text{mod } I^{m+1}F) \text{ resp. } (\text{mod } J_{m+1}F).$$

Since $\sigma_{m-1} \otimes \langle\langle -x\rangle\rangle \in P_mF$ is anisotropic, we obtain $\deg \varphi_{n,m} = \deg \psi_{n,m} = \deg' \varphi_{n,m} = \deg' \psi_{n,m} = m$.

Let now E be non-formally real and $u(E) \geq 4$. For our construction, it will again suffice to consider the case $F = E((x_1)) \cdots (((x_{n-1}))$. Since $u(E) \geq 4$ there exists an anisotropic form $\langle 1, a, b, ab \rangle \in P_2 E$.

Let $\pi_n := \langle\langle a, x_1, \ldots, x_{n-1} \rangle\rangle \in P_n F$ and $\tau := \langle -1, -a, bx_1, abx_2 \rangle$. Clearly, these two forms are anisotropic. Let $\varphi_{n,1} := (\tau \perp \pi_n)_{an}$ and $\psi_{n,1} := (\tau \perp -b\pi)_{an}$. To determine the dimension of $\varphi_{n,1}$ resp. $\psi_{n,1}$, it suffices to determine the dimensions of their residue forms corresponding to $X_0 = 1$, $X_{e_1} = x_1$, and $X_{e_2} = x_2$, as all the other residue forms of $(\tau \perp \pi_n)_{an}$ resp. $(\tau \perp -b\pi_n)_{an}$ coincide with those of π_n resp. $-b\pi_n$. We thus get

$$i_W(\tau \perp \pi_n) = i_W(\langle -1, -a, 1, a \rangle) + i_W(\langle b, 1, a \rangle) + i_W(\langle ab, 1, a \rangle) = 2$$

because $i_W(\langle -1, -a, 1, a \rangle) = 2$ and $\langle 1, a, b, ab \rangle$ is anisotropic. Similarly, $i_W(\tau \perp -b\pi_n) = 2$. This immediately implies that $\dim \varphi_{n,1} = \dim \psi_{n,1} = 2^n$. The Witt cancellation and dimension count yield $\varphi_{n,1} \perp -\psi_{n,1} \simeq \langle\langle b \rangle\rangle \otimes \pi_n \in P_{n+1} F$. Thus, $\varphi_{n,1}$ and $\psi_{n,1}$ are half-neighbors which, by construction, contained Pfister neighbors of codimension 2 of π_n. Furthermore,

$$\varphi_{n,1} \equiv \psi_{n,1} \equiv \tau \equiv \langle 1, -x_1 x_2 \rangle \qquad (\mathrm{mod}\, I^2 F) \text{ resp. } (\mathrm{mod}\, J_2 F),$$

which, since $\langle 1, -x_1 x_2 \rangle \simeq \langle\langle -x_1 x_2 \rangle\rangle$ is anisotropic, implies that $\deg \varphi_{n,1} = \deg \psi_{n,1} = \deg' \varphi_{n,1} = \deg' \psi_{n,1} = 1$.

We now show that $\varphi_{n,1}$ and $\psi_{n,1}$ are not similar. By Lemma 2.11, it suffices to show that $G(\tau) \cap -bG(\pi) = \varnothing$. So suppose $u \in G(\tau)$. By Lemma 3.1, one easily checks that then $u = 1 \in F^*/(F^*)^2$. But $1 \notin -bD(\pi) = -bG(\pi)$. In fact,

$$\langle 1 \rangle \perp b\pi \simeq \langle 1, b, ab \rangle \perp \text{ (other residue forms of } b\pi),$$

and by Springer's theorem this form is anisotropic as $\langle 1, a, b \rangle$ and the remaining residue forms of $b\pi$ are anisotropic.

Let now $m \geq 2$ and put $\sigma_{n,m} := \langle\langle x_{n-m+1}, \ldots, x_{n-1} \rangle\rangle \in P_{m-1} F$, $\varphi_{n,m} := \sigma_{n,m} \otimes \varphi_{n-m+1,1}$, and $\psi_{n,m} := \sigma_{n,m} \otimes \psi_{n-m+1,1}$. Note that $\varphi_{n-m+1,1}$ resp. $\psi_{n-m+1,1}$ are defined over $E((x_1)) \cdots ((x_{n-m}))$. It follows from Corollary 3.3 that $\varphi_{n,m}$ and $\psi_{n,m}$ are nonsimilar half-neighbors of dimension 2^n with

$$\varphi_{n,m} \equiv \psi_{n,m} \equiv \sigma_{n,m} \otimes \langle\langle -x_1 x_2 \rangle\rangle \qquad (\mathrm{mod}\, I^{m+1} F) \text{ resp. } (\mathrm{mod}\, J_{m+1} F).$$

Since $\sigma_{n,m} \otimes \langle\langle -x_1 x_2 \rangle\rangle \simeq \langle\langle -x_1 x_2, x_{n-m+1}, \ldots, x_{n-1} \rangle\rangle \in P_m F$ is anisotropic, this readily implies $\deg \varphi_{n,m} = \deg \psi_{n,m} = \deg' \varphi_{n,m} = \deg' \psi_{n,m} = m$. ∎

COROLLARY 3.5. *Let n, m be integers with $1 \leq m \leq n - 2$, let E be a non-formally real field, and let F be any field with $E(y_0, \ldots, y_n) \subset F \subset E((y_0)) \cdots ((y_n))$ (where $y_0, \ldots, y_n$ are variables). Then there exist forms $\varphi_{n,m}$ and $\psi_{n,m}$ with the same properties as in Theorem* 3.4.

Proof. This follows readily from the proof of Theorem 3.4 in the non-formally real case by putting $a = y_0$, $b = y_1$, $x_i = y_{i+1}$, $1 \leq i \leq n - 1$, and keeping the other notations there. ∎

It is an almost trivial observation that if there are two nonsimilar half-neighbors of dimension 2^n over a field F, then there must exist an anisotropic $(n+1)$-fold Pfister form over F. In particular, if F is non-formally real, one necessarily has $u(F) \geq 2^{n+1}$. This bound can in fact be reached.

COROLLARY 3.6. *Let $n \geq 3$. Then there exists a non-formally real field F with $u(F) = 2^{n+1}$ such there exist nonsimilar half-neighbors of dimension 2^n over F.*

Proof. Let $F = E((x_1)) \cdots ((x_{n-1}))$ be the iterated power series field in $n - 1$ variables over a non-formally real field E with $u(E) = 4$. By Springer's theorem, $u(F) = 2^{n-1}u(E) = 2^{n+1}$. It follows from Theorem 3.4 that F is a field with the desired properties. ∎

4. CONSTRUCTION OF NONSIMILAR HALF-NEIGHBORS OF DIMENSION 2^n AND DEGREE $n - 1$

The crucial case in constructing nonsimilar half-neighbors of dimension 2^n and degree $n - 1$ (where $n \geq 3$) will be the case $n = 3$. The higher dimensional examples are then gotten by multiplying the 8-dimensional nonsimilar half-neighbors by suitable "generic" Pfister forms in a way similar to what we did in the proof of Theorem 3.4.

Let φ and ψ be 8-dimensional half-neighbors in I^2F and let $c(\varphi)$ be the Clifford invariant of φ. It is well known that the Clifford invariant of an I^2-form of dimension $2m$ can be represented by the class of a tensor product of $m - 1$ quaternion algebras and that therefore its index is of the form 2^r for some r with $0 \leq r \leq m - 1$. In particular, the index of $c(\varphi)$ is 1, 2, 4, or 8. If the index is 1 then $\varphi \in GP_3F$, and if the index if 2 then $\varphi \in GP_{3,2}F$ (see, e.g., [Kn2, Ex. 9.12; H6, Proposition 3.11]). In these cases, we know that $\varphi \underset{hn}{\sim} \psi$ implies $\varphi \underset{sim}{\sim} \psi$ (cf. Proposition 2.8 and Corollary 2.9, see also [L1, Corollary 1] or [H5, Theorem 1.4]). However, we will construct counterexamples in the cases of index 4 as well as 8. Our constructions will be based on some deep results concerning central division algebras of exponent 2 which we recall in the sequel.

Let F be a field of characteristic $\neq 2$ and let $a \in F^*$. We put $N_F(a) = D_F(\langle 1, -a\rangle) \subset F^*$, i.e., $N_F(a)$ denotes the subset of elements of F^* represented by $\langle 1, -a\rangle$ over F. For $a, b, c \in F^*$ we define a group $N_{2,F}(a, b, c)$ as the following quotient group (compare [ELTW, 4.22]):

$$N_{2,F}(a,b,c) = \frac{N_F(a)N_F(b) \cap N_F(a)N_F(c) \cap N_F(a)N_F(bc)}{N_F(a)[N_F(a) \cap N_F(b)]}.$$

Remark 4.1. If the group $N_{2,F}(a, b, c)$ is nontrivial then $[F(\sqrt{a}, \sqrt{b}, \sqrt{c}): F] = 8$ (cf. [ELTW, Remark 4.24]), in which case a, b, c represent $\mathbf{F}_2$-linearly independent square classes of F^*.

LEMMA 4.2. (i) *If there exists an indecomposable central algebra D of index* 8 *and exponent* 2 *over F then there exist $a, b, c \in F^*$ with* $|N_{2,F}(a, b, c)| > 1$.

(ii) *To each field E of characteristic $\neq 2$ there exists a field extension F/E such that there exists an indecomposable central algebra D over F of index* 8 *and exponent* 2. *In particular, by* (i), *there exist a field extension F/E and $a, b, c \in F^*$ with $|N_{2,F}(a, b, c)| > 1$.*

As for (i), we refer to [T, Sect. 3], where a partial converse of the statement is also mentioned, namely, if there exist $a, b, c \in F^*$ with $|N_{2,F}(a, b, c)| > 1$ then there exists an indecomposable central algebra of index 8 and exponent 2 over the rational function field in three variables over F. (Note that the group denoted in [T] by $\mathcal{N}(M/F)$, where in our case $M = F(\sqrt{a}, \sqrt{b}, \sqrt{c})$, coincides with $N_{2,F}(a, b, c)$ as defined above, cf. [ELTW, 4.22].)

As for (ii), we refer to [Ka], where it is shown how one can construct indecomposable central algebras of index 8 and exponent 2 over some suitable extension of E in a "generic" way (in fact, the construction there works in any characteristic).

THEOREM 4.3. *Let E be a field such that there exist $a, b, c \in E^*$ with $|N_{2,E}(a, b, c)| > 1$ and let F be any field with $E(x, y) \subset F \subset E((x))((y))$ (where x, y are variables). Then there exist forms $\varphi, \psi \in I^2F$ of dimension* 8 *and with* $\operatorname{ind}(c(\varphi)) = 4$ *such that φ and ψ are nonsimilar half-neighbors.*

Proof. Let $a, b, c, \in E^*$ with $|N_{2,E}(a, b, c)| > 1$ and let $d \in E^*$ such that

$$d \in N_E(a)N_E(b) \cap N_E(a)N_E(c) \cap N_E(a)N_E(bc),$$
$$d \notin N_E(a)[N_E(a) \cap N_E(b)].$$

For our construction, it will suffice to consider the case $F = E((x))((y))$. Let $\alpha := \langle 1, -b\rangle \perp x\langle 1, -c\rangle \perp y\langle 1, -bc\rangle$ and $\pi := \langle\langle -a, x, y\rangle\rangle$. By

Remark 4.1 and Springer's theorem, α and π are anisotropic. Furthermore, $\alpha \in I^2F$, i.e., α is an anisotropic Albert form and we have $\mathrm{ind}(c(\alpha)) = 4$. Let now $\varphi := (\alpha \perp -\pi)_{an}$. By comparing residue forms and using Remark 4.1, one obtains

$$\varphi \simeq \langle a, -b\rangle \perp x\langle a, -c\rangle \perp y\langle a, -bc\rangle \perp xy\langle a, -1\rangle.$$

Now since $d \in N_E(a)N_E(b)$, there exist $r \in N_E(a)$ and $r' \in N_E(b)$ such that $d = rr'$. As $N_E(a) = D_E(\langle 1, -a\rangle) = G_E(\langle 1, -a\rangle)$, we get $d\langle 1, -a\rangle \simeq dr\langle 1, -a\rangle$, and similarly, $\langle 1, -b\rangle \simeq dr\langle 1, -b\rangle$ because $dr \in N_E(b)$. Hence, in WE,

$$\langle 1, -b\rangle - d\langle 1, -a\rangle = dr\langle 1, -b\rangle - dr\langle 1, -a\rangle = dr\langle a, -b\rangle.$$

In a similar way, we find $s, t \in E^*$ such that

$$\langle 1, -c\rangle - d\langle 1, -a\rangle = ds\langle a, -c\rangle,$$
$$\langle 1, -bc\rangle - d\langle 1, -a\rangle = dt\langle a, -bc\rangle.$$

We now define $\psi := (\alpha \perp -d\pi)_{an}$. Using the above, one readily checks that

$$\psi \simeq dr\langle a, -b\rangle \perp x\,ds\langle a, -c\rangle \perp y\,dt\langle a, -bc\rangle \perp xyd\langle a, -1\rangle.$$

By Proposition 2.7, $\varphi \underset{hn}{\sim} \psi$. We will show that $G_F(\alpha) \cap dG_F(\pi) = \emptyset$ which, by Lemma 2.11, implies that φ is not similar to ψ. Let $u \in G_F(\alpha)$. By Lemma 3.1 and by considering the residue forms, it follows that there exists $v \in G_E(\langle 1, -b\rangle) \cap G_E(\langle 1, -c\rangle) = N_E(b) \cap N_E(c) \subset E^*$ such that $u = v \in F^*/(F^*)^2$. Suppose $G_F(\alpha) \cap dG_F(\pi) \neq \emptyset$. Then, by the above, there exists $v \in N_E(b) \cap N_E(c)$ such that $v \in dG_F(\pi) = dD_F(\pi)$, i.e., $\langle v\rangle \perp -d\pi$ is isotropic. Considering the residue forms of this form, we necessarily have that, over E, $\langle v, -d, da\rangle$ is isotropic, i.e., $v \in dD_E(\langle 1, -a\rangle) = dN_E(a)$. Hence, $dv \in N_E(a)$ and thus $(dv)v \in N_E(a)[N_E(b) \cap N_E(c)]$ which obviously implies $d \in N_E(a)[N_E(b) \cap N_E(c)]$, a contradiction.

Finally, note that $\varphi \equiv \alpha \pmod{I^3F}$ and that therefore $c(\varphi) = c(\alpha) \in \mathrm{Br}_2 F$. In particular, $\mathrm{ind}(c(\varphi)) = 4$. ∎

COROLLARY 4.4. *Let $n \geq 3$. Then there exists a field F with forms φ and ψ of dimension 2^n such that φ and ψ are nonsimilar half-neighbors and* $\deg \varphi = \deg \psi = \deg' \varphi = \deg' \psi = n - 1$.

Proof. Let F_0 be a field as in Theorem 4.3 such that there exist forms $\varphi_0, \psi_0 \in I^2F_0$ of dimension 8 which are nonsimilar half-neighbors. If $n = 3$ we are done. If $n \geq 4$ let F be any field with $F_0(x_1, \dots, x_{n-3}) \subset F \subset F_0((x_1)\cdots((x_{n-3}))$ (where the x_i are variables). Let $\sigma := \langle\langle x_1, \dots, x_{n-3}\rangle\rangle$,

$\varphi := \sigma \otimes \varphi_0$, and $\psi := \sigma \otimes \psi_0$. It is rather obvious that $\deg \varphi = \deg \psi = \deg' \varphi = \deg' \psi = n - 1$ and that φ and ψ are half-neighbors of dimension 2^n. They are nonsimilar by Corollary 3.3. ∎

Our next aim is to construct nonsimilar half-neighbors of dimension 2^n, $n \geq 3$, which are in I^2F such that the index of their Clifford invariant has a certain prescribed value 2^s. By the remarks at the beginning of this section, we know that if $n = 3$ then the only possible values are $s = 2$ (such examples have been constructed in Theorem 4.3) and $s = 3$. If $n \geq 4$ we know that necessarily $0 \leq s \leq 2^{n-1} - 1$, and none of these values has been ruled out. In fact, examples with $s = 0$ are given in Corollary 4.4 as the nonsimilar half-neighbors constructed there are in $I^{n-1}F \subset I^3F$ and have therefore trivial Clifford invariant. We will now produce examples for all the remaining values of s.

THEOREM 4.5. *Let $n \geq 3$. Let $\tilde{\varphi}$ and $\tilde{\psi}$ be forms over a field E which are nonsimilar half-neighbors of dimension 2^n. Suppose $\tilde{\varphi}, \tilde{\psi} \in I^2E$ and* $\mathrm{ind}(c(\varphi)) = 2^r$ *for some $0 \leq r \leq 2^{n-1} - 1$ ($r \geq 2$ if $n = 3$). Let D denote the division algebra with $c(\tilde{\varphi}) = [D] \in \mathrm{Br}_2 E$ and let s be any integer with $r \leq s \leq 2^{n-1} - 1$. Then there exist a field extension F/E and $t = s - r$ quaternion algebras $Q_1, \dots, Q_t$ over F such that $A := D_F \otimes Q_1 \otimes \cdots \otimes Q_t$ is a division algebra over F, in particular, $\mathrm{ind}_R A = 2^s$, and there exist nonsimilar half-neighbors φ, ψ over F of dimension 2^n which are in I^2F and such that $c(\varphi) = [A] \in \mathrm{Br}_2 F$. In particular, $\mathrm{ind}_F(c(\varphi)) = 2^s$.*

COROLLARY 4.6. *Let $n \geq 3$ and let s be an integer with $s \in \{2, 3\}$ if $n = 3$ and $0 \leq s \leq 2^{n-1} - 1$ if $n \geq 4$. Then there exists a field F with nonsimilar half-neighbors φ and ψ of dimension 2^n such that $\varphi, \psi \in I^2F$ and* $\mathrm{ind}(c(\varphi)) = 2^s$.

Proof of Corollary 4.6. In view of Theorem 4.5, it suffices to find a field E with nonsimilar half-neighbors of dimension 2^n which are in I^2E and have Clifford invariant index $4 = 2^2$ if $n = 3$, or which are in I^3E (which implies that the Clifford invariant index is $1 = 2^0$) if $n \geq 4$, respectively. Such E exists by Theorem 4.3 and Corollary 4.4. ∎

Proof of Theorem 4.5. If $t = 0$ there is nothing to show. It thus suffices to consider the case $t = 1$ as the general case follows by a repeated application of the case $t = 1$.

So let $\tilde{\varphi}$ and $\tilde{\psi}$ be nonsimilar half-neighbors over a field E such that $\tilde{\varphi}, \tilde{\psi} \in I^2E$ and $\mathrm{ind}_E(c(\varphi)) = 2^r$ with $r < 2^{n-1} - 1$. Let D be the division algebra over E with $c(\tilde{\varphi}) = [D] \in \mathrm{Br}_2 E$. After scaling, we may assume $\tilde{\varphi} \perp \tilde{\psi} \in GP_{n+1}E$. Note that obviously also $c(\tilde{\psi}) = [D]$ as $\tilde{\varphi} \equiv \tilde{\psi}$ (mod I^3F). Let $K = E(x, y)$ be the rational function field in the variables x, y over E. Then $A := D_K \otimes (x, y)_K$ is clearly a division algebra over K and we have $\mathrm{ind}_K A = 2(\mathrm{ind}_E D) = 2^{r+1} = 2^s$.

Let $\varphi' := \tilde{\varphi}_K \perp \langle\langle -x, -y\rangle\rangle$ and $\psi' := \psi'_K \perp \langle\langle -x, -y\rangle\rangle$. These are forms in I^2K with $c(\varphi') = c(\psi') = c(\tilde{\varphi}_K)c(\langle\langle -x, -y\rangle\rangle) = [D_K][(x, y)_K] = [A]$ in $\mathrm{Br}_2 K$.

Claim 1. There exists a field L_1 in the generic splitting tower φ' over K such that $\mathrm{ind}_{L_1} A_{L_1} = \mathrm{ind}_K A = 2^s$ and $\dim(\psi'_{L_1})_{an} \geq \dim(\varphi'_{L_1})_{an} = 2^n$.

Proof of Claim 1. Let $K' = K(\sqrt{x})$. Then K'/E is purely transcendental and thus $\tilde{\varphi}_{K'}$ and $\tilde{\psi}_{K'}$ are both anisotropic. However, $\langle\langle -x, -y\rangle\rangle$ is isotropic and hence hyperbolic over K'. Thus, $(\varphi'_{K'})_{an} \simeq \tilde{\varphi}_{K'}$ and $(\psi'_{K'})_{an} \simeq \tilde{\psi}_{K'}$, and these are forms of dimension 2^n. Therefore there exists a field L_1 in the generic splitting tower of φ' over K such that $\dim(\varphi'_{L_1})_{an} = 2^n$. Furthermore, by [Kn1, Corollary 3.9 and Proposition 5.13], $L_1 \cdot K'/K'$ is purely transcendental. Hence, the anisotropic form $(\psi'_{K'})_{an}$ stays anisotropic over $L_1 \cdot K'$ and thus $\dim(\psi'_{L_1})_{an} \geq \dim(\psi'_{L_1 \cdot K'})_{an} = \dim(\psi'_{K'})_{an} = 2^n$.

Now L_1/K is obtained by successively taking function fields of forms of dimension $\geq 2^n + 2$. Since $\mathrm{ind}_K A = 2^s$ with $s \leq 2^{n-1} - 1$, it follows from Merkurjev's index reduction theorems [M] that $\mathrm{ind}_{L_1} A_{L_1} = \mathrm{ind}_K A = 2^s$.

Claim 2. There exists a field L_2 in the generic splitting tower ψ'_{L_1} over L_1 such that $\mathrm{ind}_{L_2} A_{L_2} = \mathrm{ind}_K A = 2^s$ and $\dim(\psi'_{L_2})_{an} = \dim(\varphi'_{L_2})_{an} = 2^n$.

Proof of Claim 2. If $\dim(\psi'_{L_1})_{an} = 2^n$ we are done by putting $L_2 = L_1$. Otherwise, since $L_1 \subset L_1 \cdot K'$ and $\dim(\psi'_{L_1 \cdot K'})_{an} = 2^n$ by the above, there exists such a field L_2 in the generic splitting tower of ψ'_{L_1}, and it is obtained by successively taking function fields of forms of dimension $\geq 2^n + 2$. As before, this yields $\mathrm{ind}_{L_2} A_{L_2} = \mathrm{ind}_K A = 2^s$. Furthermore, the anisotropic form $(\varphi'_{L_1})_{an}$ of dimension 2^n will stay anisotropic over L_2 by [H3, Theorem 1].

Claim 3. $L_2 \cdot (L_1 \cdot K')/E$ is purely transcendental.

Proof of Claim 3. Since $\dim(\psi'_{L_1 \cdot K'})_{an} = \dim(\psi'_{L_2})_{an} = 2^n$ and since L_2 is a field in the generic splitting tower of ψ'_{L_1} and $L_1 \subset L_1 \cdot K'$, we have that $L_2 \cdot (L_1 \cdot K')/L_1 \cdot K'$ is purely transcendental. We have already seen that $L_1 \cdot K'/K'$ and K'/E are purely transcendental. Hence, $L_2 \cdot (L_1 \cdot K')/E$ is purely transcendental.

We now put $F = L_2$ and define forms $\varphi := (\varphi'_F)_{an}$ and $\psi := (\psi'_F)_{an}$ over F which are of dimension 2^n by the above. By construction, we have $\varphi, \psi \in I^2F$ and $\mathrm{ind}_F(c(\varphi)) = \mathrm{ind}_F(c(\psi)) = 2^s$. It remains to show that φ and ψ are nonsimilar half-neighbors.

In WF we have

$$\varphi - \psi = (\varphi' - \psi')_F = \left(\tilde{\varphi} - \tilde{\psi}\right)_F.$$

But $\tilde{\varphi} \perp -\tilde{\psi} \in GP_{n+1}E$ and by comparing dimensions we obtain $\varphi \perp -\psi \simeq (\tilde{\varphi} \perp -\tilde{\psi})_F \in GP_{n+1}F$ which implies that φ and ψ are half-neighbors.

Let $F' = L_2 \cdot (L_1 \cdot K')$. By Claim 3, F'/E is purely transcendental. We also have $\varphi_{F'} \simeq \tilde{\varphi}_{F'}$ and $\psi_{F'} \simeq \tilde{\psi}_{F'}$. By Corollary 3.2, $\tilde{\varphi}$ and $\tilde{\psi}$ don't become similar over F' as they are not similar over E. Hence, φ and ψ are not similar over F because $F \subset F'$ and they are not similar over F'. ∎

5. FIELDS OVER WHICH HALF-NEIGHBORS ARE SIMILAR

We have seen in the previous sections that over fields of sufficient complexity half-neighbors need not be similar. On the other hand, over relatively "simple" fields one would expect that half-neighbors are always similar.

EXAMPLE 5.1. Let F be a non-formally real field such that there are no anisotropic 4-fold Pfister forms, or, equivalently, such that $I^4F = 0$. Then half-neighbors over F are always similar. In fact, half-neighbors of dimension 2^n with $n \leq 2$ are always similar (independent of F), and they are similar for $n \geq 3$ because nonsimilar half-neighbors of dimension 2^n can only exist if there exist anisotropic $(n+1)$-fold Pfister forms. In particular, if F is non-formally real and $u(F) < 16$ then half-neighbors are always similar.

The main results in this section will concern fields with small Hasse number and m-linked fields. Let m be an integer ≥ 2. A field is called m-linked if to each $\varphi \in I^mF$ there exists some $\pi \in P_mF$ such that $\varphi \equiv \pi \pmod{I^{m+1}F}$. Equivalently, F is m-linked if to any $\pi_1, \pi_2 \in P_mF$ there exist $x_1, x_2 \in F^*$ and $\tau \in P_{m-1}F$ such that $\pi_i \simeq \tau \otimes \langle\langle x_i \rangle\rangle$, $i = 1, 2$. If F is m-linked then F is n-linked for all $n \geq m$, and in the case $m = 2$ we simply say that F is linked. This linkage property was first studied in [EL] where it was shown that if F is linked then $u(F) \in \{0, 1, 2, 4, 8\}$ and that all these values are possible.

LEMMA 5.2. *Let F be a field and $m \geq 2$. Then the following are equivalent.*

(i) *F is m-linked;*

(ii) *For all anisotropic $\varphi \in I^mF$ there exist $\tau \in P_{m-1}F$ and an even-dimensional form σ over F such that $\varphi \simeq \tau \otimes \sigma$;*

(iii) *For all $\varphi \in I^mF$ with* $\dim \varphi < 2^{n+1}$ *there exist forms $\pi_i \in GP_iF$, $m \leq i \leq n$, such that $\varphi = \sum_{i=m}^{n} \pi_i$ in WF.*

If any of these conditions holds, then $I_t^{m+2}F = 0$.

Proof. The equivalence of (i) and (ii) has been shown in [H4]. That (iii) implies (i) is rather obvious and left to the reader.

(ii) $\Rightarrow$ (iii). Let $\varphi \in I^m F$ with $\dim \varphi < 2^{n+1}$. We may assume that φ is anisotropic. Part (ii) then readily implies that $\dim \varphi \equiv 0 \pmod{2^m}$, and by the Arason–Pfister Hauptsatz we have $n \geq m$. We proceed by induction on $n - m$. If $n - m = 0$ then, by the above, $\dim \varphi = 2^m$ and thus $\varphi \in GP_m F$ as $\varphi \in I^m F$ (cf., e.g., [AP, Kor. 3]). Clearly, we are done in this case. So suppose $n - m > 0$. Since we know that (ii) implies that F is m-linked, there exists $\pi_m \in GP_m F$ such that $\varphi \equiv \pi_m \pmod{I^{m+1}F}$. After scaling π_m (which doesn't affect the above equivalence modulo $I^{m+1}F$), we may assume that $D(\pi_m) \cap D(\varphi) \neq \varnothing$ which then implies that $\varphi \perp -\pi_m$ is isotropic. Let $\psi := (\varphi \perp -\pi_m)_{an} \in I^{m+1}F$. We then have $\dim \psi < \dim \varphi + \dim \pi_m = \dim \varphi + 2^m$. On the other hand, since $\dim \varphi \equiv 0 \pmod{2^m}$ and $\dim \varphi < 2^{n+1}$, we have $\dim \varphi \leq 2^{n+1} - 2^m$ and hence $\dim \psi < 2^{n+1}$. Now $\psi \in I^{m+1}F$, F is $(m+1)$-linked, and $n - (m+1) < n - m$. By the induction hypothesis, $\psi = \sum_{i=m+1}^{n} \pi_i$ in WF for some $\pi_i \in GP_i F$, $m + 1 \leq i \leq n$. Hence, $\varphi = \psi + \pi_m = \sum_{i=m}^{n} \pi_i$ in WF as desired.

The fact that m-linkage implies $I_t^{m+2}F = 0$ was shown in [EL, Corollary 2.8]. ∎

Remark 5.3. In [E, Lemma 4.4], it was shown that (i) is equivalent to the fact that each $\varphi \in I^m F$ can be written as $\varphi = \sum_{i=m}^{t} \pi_i$ in WF for some $t \geq m$ and suitable $\pi_i \in GP_i F$, $m \leq i \leq t$. Our statement in (iii) is more precise as it gives us an upper bound for the minimal possible such t in terms of the dimension of φ. This bound, however, will be crucial in some of the later proofs.

In general, this bound cannot be improved. Suppose that F is m-linked, $m \geq 2$, and that $\varphi \in I^m F$ is an anisotropic form with $2^n \leq \dim \varphi < 2^{n+1}$ where $n \geq m$. Let us write $\varphi \in \sum_{i=m}^{n} \pi_i$ in WF with $\pi_i \in GP_i F$, $m \leq i \leq n$. Then, by adding up dimensions and using the fact that φ is anisotropic, we see that $\pi_n \in GP_n F$ cannot be hyperbolic and is therefore anisotropic.

THEOREM 5.4. *Let F be a field with $\tilde{u}(F) < 2^n$, $n \geq 3$. Let $n - 2 \geq m \geq 1$ and suppose that $I_t^{m+2}F = 0$. Let φ and ψ be half-neighbors over F of dimension 2^n and suppose there exists $\pi_m \in P_m F$ such that $\varphi \equiv \pi_m$ $\pmod{I^{m+1}F}$. Then $\varphi \underset{sim}{\sim} \psi$.*

Proof. We may assume that φ and ψ are anisotropic and thus, necessarily, that F is formally real. Using the fact that F is SAP, we may also assume that, after scaling, $\operatorname{sgn}_P \varphi \geq 0$ for all $P \in X_F$.

Since F is SAP, there exists $\pi_n \in P_n F$ such that $\operatorname{sgn}_P \pi_n = 2^n$ for all $P \in X_F$ with $\operatorname{sgn}_P \varphi = 2^n$, and $\operatorname{sgn}_P \pi_n = 0$ for all other P. Let $\tau := (\varphi \perp -\pi_n)_{an}$. Then $|\operatorname{sgn}_P \tau| < 2^n$ for all $P \in X_F$ and we get $\dim \tau < 2^n$ as we also have $\tilde{u}(F) < 2^n$. Now $\varphi = \tau + \pi_n$ in WF. Since $\dim \tau < 2^n$ and

$\pi_n \in P_n F$, we can apply Proposition 2.7 (and its proof) to conclude that, after scaling, $\psi = \tau + a\pi_n$ in WF for some $a \in F^*$. We will prove that $\varphi \underset{sim}{\sim} \psi$ by showing that $G(\tau) \cap aG(\pi_n) \neq \emptyset$ and then applying Lemma 2.11.

Let $Z := \{P \in X_F \mid \operatorname{sgn}_P \pi_n = 2^n$ and $a <_P 0\}$. Let $P \in Z$. Suppose $\operatorname{sgn}_P \tau > 0$. Then $\operatorname{sgn}_P \varphi = \operatorname{sgn}_P \tau + \operatorname{sgn}_P \pi_n > 2^n = \dim \varphi$, a contradiction. Suppose $\operatorname{sgn}_P \tau < 0$. Then $\operatorname{sgn}_P \psi = \operatorname{sgn}_P \tau + \operatorname{sgn}_P a\pi_n = \operatorname{sgn}_P \tau - \operatorname{sgn}_P \pi_n < -2^n = -\dim \psi$, again a contradiction. Hence, for all $P \in Z$ we have $\operatorname{sgn}_P \tau = 0$. Now $\varphi \equiv \tau \equiv \pi_m \pmod{I^{m+1}F}$. Thus, $\tau = \pi_m + \beta$ in WF for some $\beta \in I^{m+1}F$. We have $\operatorname{sgn}_P \tau = \operatorname{sgn}_P \pi_m + \operatorname{sgn}_P \beta$, and since $\operatorname{sgn}_P \pi_m \in \{0, 2^m\}$ and $\operatorname{sgn}_P \beta \equiv 0 \pmod{2^{m+1}}$, we get $\operatorname{sgn}_P \pi_m = \operatorname{sgn}_P \beta = 0$ for all $P \in Z$.

Let $y' \in F^*$ such that $y' <_P 0$ for all $P \in Z$, and $y' >_P 0$ for all $P \in X_F \setminus Z$. Let

$$\mu := \pi_m \perp -y'\langle \underbrace{1, \dots, 1}_{2^n - 2^m + 1} \rangle .$$

Since $\dim \mu = 2^n + 1$ and since $\tilde{u}(F) < 2^n$, it is well know that μ is a Pfister neighbor (see, e.g., [F2, Proposition 4.7]). If $P \in Z$ then π_m is indefinite at P by the above, and thus also μ, and if $P \in X_F \setminus Z$ then $y' >_P 0$ and μ is indefinite at P because $\langle 1, -y' \rangle \subset \pi_m \perp \langle -y' \rangle \subset \mu$. Hence, μ in totally indefinite and its associated $(n+1)$-fold Pfister form is therefore torsion and hence hyperbolic as $I_t^{n+1}F = 0$. In particular, μ is isotropic and there exists some $y \in D(y'\langle 1, \dots, 1 \rangle) \cap D(\pi_m)$. Note that yy' is a sum of squares so that $y <_P 0$ iff $y' <_P 0$ iff $P \in Z$. Note also that $y \in D(\pi_m) = G(\pi_m)$.

By the above, $\operatorname{sgn}_P \beta = 0$ for all $P \in Z$, and $\operatorname{sgn}_P \langle\langle -y \rangle\rangle = 0$ for all $P \in X_F \setminus Z$. Hence, $\operatorname{sgn}_P(\beta \otimes \langle\langle -y \rangle\rangle) = 0$ for all $P \in X_F$ which implies that $\beta \otimes \langle\langle -y \rangle\rangle \in I_t^{m+2}F = 0$. Hence, $\beta \otimes \langle\langle -y \rangle\rangle$ is hyperbolic and $y \in G(\beta)$.

If $P \in Z$ then, by definition of Z and construction of y, we have $y <_P 0$ and $a <_P 0$, hence $ay >_P 0$ and $\operatorname{sgn}_P \langle\langle -ay \rangle\rangle = 0$. If $P \in X_F \setminus Z$, then $y >_P 0$ and either π_n is definite at P, in which case necessarily $a >_P 0$ and again $\operatorname{sgn}_P \langle\langle -ay \rangle\rangle = 0$, or $\operatorname{sgn}_P \pi_n = 0$. In any case, we find that $\operatorname{sgn}_P(\pi_n \otimes \langle\langle -ay \rangle\rangle) = 0$ for all $P \in X_F$ and thus $\pi_n \otimes \langle\langle -ay \rangle\rangle \in I_t^{n+1}F = 0$. Similarly as above, we get $ay \in G(\pi_n)$ or $y \in aG(\pi_n)$.

By Lemma 2.11 we have $G(\tau) = G(\pi_m) \cap G(\beta)$, and the above shows that $y \in G(\tau) \cap aG(\pi_n)$, which again by Lemma 2.11 implies that $\varphi \simeq y\psi$. ∎

COROLLARY 5.5. *Let F be a field with $\tilde{u}(F) \leq 6$. If φ and ψ are forms over F which are half-neighbors, then they are similar.*

Proof. It suffices to prove this for anisotropic φ, ψ of dimension 2^n with $n \geq 3$. In this case, $2^n > \tilde{u}(F)$, and clearly $I_t^3 F = 0$ as $\tilde{u}(F) \leq 6$. The claim follows readily by applying Theorem 5.4 for $m = 1$ because obviously $\varphi \equiv \langle\langle -d \rangle\rangle \pmod{I^2 F}$ where $d = d_{\pm}\varphi$. ∎

The previous corollary shows that over global fields ($\tilde{u}(F) = 4$) or fields of transcendence degree ≤ 1 over the reals ($\tilde{u}(F) \leq 2$), half-neighbors are always similar, but that there are also non-linked fields (e.g., if $\tilde{u}(F) = 6$) over which half-neighbors are always similar.

We are able to show a result similar to Theorem 5.4 after replacing the condition $\tilde{u}(F) < 2^n$ by F being m-linked ($m \geq 2$). Note that generally m-linked fields need not have finite Hasse number. For example, one readily sees that $\mathbf{Q}((t))$ is 4-linked but $\tilde{u}(\mathbf{Q}((t))) = \infty$.

THEOREM 5.6. *Let $n \geq m \geq 2$ and let F be an m-linked field. Let φ and ψ be half-neighbors over F of dimension 2^n which are in $I^m F$. Then φ and ψ are similar.*

Proof. We will keep the notations as close as possible to those in the proof of Theorem 5.4 since many of the arguments will parallel those of the proof there. It should also be noted that the condition $I_t^{m+2} F = 0$ stated in Theorem 5.4 is automatically fulfilled here as F is assumed m-linked (see Lemma 5.2).

We may assume that φ is anisotropic. By Lemma 5.2, we can write $\varphi = \sum_{i=m}^{n} \pi_i$ in WF with $\pi_i \in GP_i F$. Since φ is anisotropic of dimension 2^n, we have by Remark 5.3 that π_n is anisotropic. Thus, if $m \in \{n, n-1\}$ then $\varphi \in GP_n F \cup GP_{n,n-1} F$, and we have similarity by Proposition 2.8 and Corollary 2.9.

If F is non-formally real, then $I_t^{m+2} F = I^{m+2} F = 0$ and thus necessarily $n \leq m + 1$ as π_n is anisotropic. In this case we are done by the above. So let us assume for the remainder of the proof that F is formally real and that $n \geq m + 2$.

After scaling, we may assume that $\pi_n \in P_n F$. Let $\tau := \perp_{i=m}^{n-1} \pi_i$ and $\beta := \perp_{i=m+1}^{n-1} \pi_i$. Note that $\tau = \pi_m + \beta$ with $\pi_m \in GP_m F$ and $\beta \in I^{m+1} F$. Comparing dimensions yields $\dim \tau < 2^n$. We have $\varphi = \tau + \pi_n$ in WF, and by Proposition 2.7 (and its proof), we therefore get, after scaling, that $\psi = \tau + a\pi_n$ in WF for some $a \in F^*$. As in the proof of Theorem 5.4, we define $Z = \{P \in X_F \mid \operatorname{sgn}_P \pi_n = 2^n \text{ and } a <_P 0\}$ and get that $\operatorname{sgn}_P \tau = \operatorname{sgn}_P \pi_m = \operatorname{sgn}_P \beta = 0$ for all $P \in Z$. As F is SAP because F is m-linked, we can find as before a $y' \in F^*$ with $y' <_P 0$ for all $P \in Z$ and $y' >_P 0$ otherwise.

The remainder of the proof runs now exactly in the same manner as in Theorem 5.4, provided we can find some $y \in G(\pi_m)$ with yy' a sum of squares (i.e., $y <_P 0$ iff $y' <_P 0$ iff $P \in Z$). Let $\rho \in P_m F$ such that

$\pi_m \underset{sim}{\sim} \rho$. Then $D(\rho) = G(\rho) = G(\pi_m)$ and $\mathrm{sgn}_P \rho = \mathrm{sgn}_P \pi_m = 0$ for all $P \in Z$. This time we consider

$$\mu := \rho \perp -y'\Big(\langle\langle \underbrace{1,\dots,1}_{m} \rangle\rangle \perp \langle 1 \rangle\Big),$$

and the existence of the desired y is assured if we can show that μ is isotropic.

Since F is m-linked, there exist $\sigma \in P_{m-1}F$ and $s, t \in F^*$ such that $\rho \simeq \sigma \otimes \langle\langle s \rangle\rangle$ and $\langle\langle 1,\dots,1 \rangle\rangle \simeq \sigma \otimes \langle\langle t \rangle\rangle$. Hence,

$$\mu \subset \rho \perp -y'\langle\langle 1,\dots,1 \rangle\rangle \perp -y'\sigma \simeq \sigma \otimes \langle 1, s, -y', -y't, -y' \rangle.$$

Now $\alpha := \langle 1, s, -y', -y't, -y', y'st \rangle \in I^2F$ and we have $\mu \subset \sigma \otimes \alpha \in I^{m+1}F$. Since F is $(m+1)$-linked, it follows from Lemma 5.2 that $\dim(\sigma \otimes \alpha)_{an}$ is divisible by 2^{m+1}. On the other hand, $\dim(\sigma \otimes \alpha) = 3 \cdot 2^m$. Since $(\sigma \otimes \alpha)_{an}$ is divisible by σ it follows readily that there exists a form η of dimension 4 such that $\sigma \otimes \alpha = \sigma \otimes \eta$ in WF. Furthermore, since $\sigma \otimes \eta \in I^{m+1}F$ is of dimension 2^{m+1} we have that $\sigma \otimes \eta \in GP_{m+1}F$, so that in fact we may assume that $\eta \in GP_2F$ as $\sigma \in P_{m-1}F$. Using Witt cancellation and comparing dimensions, we get

$$\mu \subset \sigma \otimes \langle 1, s, -y', -y't, -y' \rangle \simeq \sigma \otimes (\eta \perp \langle -y'st \rangle).$$

Now $\eta \in GP_2F$ and thus $\eta \perp \langle -y'st \rangle$ is a Pfister neighbor, say, of $\gamma \in P_3F$. Let $\chi := \sigma \otimes \gamma \in GP_{m+2}F$. The above shows that μ is similar to a subform of γ, and since $\dim \mu = 2^{m+1} + 1$ we have that μ is a Pfister neighbor of χ.

Note that we have $\langle 1, -y' \rangle \subset \rho \perp \langle -y' \rangle \subset \mu$. If $P \in Z$ then $\mathrm{sgn}_P \rho = 0$ and ρ and hence also μ is indefinite at P. If $P \in X_F \setminus Z$ then $y' >_P 0$, in which case $\langle 1, -y' \rangle$ is indefinite at P and thus also μ. We conclude that μ is indefinite at all P, and therefore χ is also totally indefinite. This implies that χ is a torsion $(m+2)$-fold Pfister form. But $I_t^{m+2}F = 0$ as F is m-linked. Hence, χ is hyperbolic and its Pfister neighbor μ is isotropic. ∎

6. EQUIVALENCE RELATIONS FOR QUADRATIC FORMS

In this section, we will give an overview of some of the known results, concerning the equivalence relations defined for quadratic forms by similarity "$\underset{sim}{\sim}$," birational equivalence "$\underset{bir}{\sim}$," stably birational equivalence "$\underset{stb}{\sim}$," and the equivalence "$\underset{hn}{\sim}$" defined by two forms being half-neighbors. Of particular interest will be the question of how these relations relate to each other. We will only consider the restriction of the above equivalence relations to the set of forms of the same given dimension, where we obviously have to assume the dimension to be a 2-power in the case of half-neighbors. (Among these four relations, only " $\underset{stb}{\sim}$ " allows the possibility of forms of different dimension to be equivalent.) The reader might be easily convinced that the case of isotropic forms will not yield any interesting results in our context. Thus, we may assume that the forms we consider are anisotropic.

THEOREM 6.1. *Let φ and ψ be anisotropic forms over F with* $\dim \varphi = \dim \psi = m$.

(i) $\varphi \underset{sim}{\sim} \psi \Rightarrow \varphi \underset{bir}{\sim} \psi \Rightarrow \varphi \underset{stb}{\sim} \psi$.

(ii) *If $m = 2^n$ then* $\varphi \underset{sim}{\sim} \psi \Rightarrow \varphi \underset{hn}{\sim} \psi \Rightarrow \varphi \underset{stb}{\sim} \psi$.

(iii) *If $m \leq 4$ then* $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{bir}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$.

(iv) *If $m = 5$ or 6 and φ is not a Pfister neighbor, then* $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{bir}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$.

(v) *Let $n \geq 3$ and φ be a Pfister neighbor of an n-fold Pfister form. If $2^n - 1 \leq m \leq 2^n$ then* $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{bir}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$. *If $2^n - n \leq m \leq 2^n$ then* $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{bir}{\sim} \psi$. *If $2^{n-1} + 1 \leq m \leq 2^n - 2$ then generally* $\varphi \underset{bir}{\sim} \psi \not\Rightarrow \varphi \underset{sim}{\sim} \psi$.

(vi) *Let $r = \dim(\varphi_{F(\varphi)})_{an}$. If $r \leq 4$ then* $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{bir}{\sim} \psi$. *If $r \leq 1$ then* $\varphi \underset{bir}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$. *If $r \geq 2$ then generally* $\varphi \underset{bir}{\sim} \psi \not\Rightarrow \varphi \underset{sim}{\sim} \psi$.

(vii) *Let $\varphi \in GP_{n,l}F$, $1 \leq l < n$. Then* $\varphi \underset{hn}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$. *If $l = n - 1$ then* $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$ (*in particular,* $\varphi \underset{stb}{\sim} \psi \Leftrightarrow \varphi \underset{bir}{\sim} \psi \Leftrightarrow \varphi \underset{hn}{\sim} \psi \Leftrightarrow \varphi \underset{sim}{\sim} \psi$). *If $1 \leq l \leq n - 2$ then generally* $\varphi \underset{stb}{\sim} \psi \not\Rightarrow \varphi \underset{hn}{\sim} \psi$.

(viii) *If $n \geq 3$ and $m = 2^n$, then generally* $\varphi \underset{stb}{\sim} \psi \not\Rightarrow \varphi \underset{hn}{\sim} \psi \not\Rightarrow \varphi \underset{sim}{\sim} \psi$.

(ix) *If $m = 2^n$, $n \geq 3$, and φ contains a Pfister neighbor of codimension ≤ 1 of some n-fold Pfister form, then* $\varphi \underset{hn}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$. *However, if φ contains only a Pfister neighbor of codimension 2 but not of codimension 1 of some n-fold Pfister form, then generally* $\varphi \underset{hn}{\sim} \psi \not\Rightarrow \varphi \underset{sim}{\sim} \psi$.

(x) *If $m = 8$ and $\varphi \in I^2F$ then* $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{hn}{\sim} \psi$. *If* $\operatorname{ind} c(\varphi) = 1$ *or 2 then* $\varphi \underset{hn}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$. *If* $\operatorname{ind} c(\varphi) = 4$ *or 8 then generally* $\varphi \underset{hn}{\sim} \psi \not\Rightarrow \varphi \underset{sim}{\sim} \psi$.

Proof. Part (i) is trivial and (ii) has been shown in Corollary 2.6.

In (iii), $\varphi \underset{bir}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$ has been proved in [W], but the proof there can easily be modified to yield $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$ (see also [O]).

In (iv), it obviously suffices to show $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$. The case $m = 5$ was shown in [H2]. In the case $m = 6$ and φ an Albert form, the result is due to Leep [Le]. If $m = 6$ and $d = d_{\pm}\varphi \neq 1$, then if φ is isotropic over $L = F(\sqrt{d})$ (but not hyperbolic as φ is not a Pfister neighbor), the result follows readily from [H1, Theorem 3], and if φ_L stays anisotropic then this was shown by Laghribi [L3].

(v) Let φ be a Pfister neighbor of the n-fold Pfister form π. $\varphi \underset{stb}{\sim} \psi$ implies that ψ is also a Pfister neighbor of π (cf. [H3, Proposition 2]). If $m = 2^n - 1$ or 2^n then $\varphi \underset{sim}{\sim} \psi$ (cf. [W, Theorem 3.4]; [Kn1, Theorem 4.2]). If $2^n - n \leq m \leq 2^n$ then $\varphi \underset{bir}{\sim} \psi$ by [AhO, Corollary 2.5 and Theorem 1.6]. Let now $m = 2^{n-1} + l$ where $1 \leq l \leq 2^{n-1} - 2$ and let E be any field with an anisotropic $\tau \in P_{n-1}F$, and let us write $\tau \simeq \sigma \otimes \langle\langle s \rangle\rangle$ with $\sigma \in P_{n-2}F$ and $s \in E^*$. Let $F = E(x)$ be the rational function field in the variable x over E. Let $r_1 = \min(l - 1, 2^{n-2} - 1\}$. Let $r_2 = l - r_1$. Then $r_i \leq 2^{n-2} - 1$, $i = 1, 2$, and $2^{n-2} + r_1 \notin \{2^{n-1}, l\}$. Let now $\sigma_i \subset \sigma$ (over E) with $\dim \sigma_i = r_i$, $i = 1, 2$, and let $\mu \subset \tau$ (over E) with $\dim \mu = l$. Let $\varphi := \tau \perp x\mu$ and $\psi := \sigma \otimes \langle\langle x \rangle\rangle \perp s\sigma_1 \perp xs\sigma_2$. Obviously, φ and ψ are subforms of $\tau \otimes \langle\langle x \rangle\rangle \simeq \sigma \otimes \langle\langle x \rangle\rangle \otimes \langle\langle s \rangle\rangle \in P_nF$. After passing to $E((x))$ and comparing dimensions of the residue forms, one readily sees by Lemma 3.1 that φ and ψ are not similar. However, by [AhO, Theorem 1.6], $\varphi \underset{bir}{\sim} \psi$.

(vi) Let $r = \dim((\varphi_{F(\varphi)})_{an} \leq 4$. Suppose first that $\dim \varphi \leq 8$. By [H7, Theorem 4.1], this implies that $\dim \varphi \leq 6$, or φ is a Pfister neighbor of some 3-fold Pfister form, or $\varphi \in GP_{3,2}F$. The fact that in these cases $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{bir}{\sim} \psi$ follows now from (iii), (iv), (v), and from (vii) below. Finally, suppose $\dim \varphi > 8$. In this case, it follows from [K, Theorem 1, Proposition 1] that $(\varphi_{F(\varphi)})_{an}$ is defined over F. By [Kn2, Theorem 7.13], φ is a Pfister neighbor of some n-fold Pfister form, $n \geq 4$, of codimension $r \leq 4$, and the claim follows from (v).

If $r \leq 1$ then φ is a Pfister neighbor of codimension r and $\varphi \underset{bir}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$ follows from (v) as well. As counterexamples in the case $r \geq 2$ one can take those constructed in (v).

(vii) $\varphi \underset{hn}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$ has been shown in Proposition 2.8. The fact that $\varphi \underset{stb}{\sim} \psi \Rightarrow \varphi \underset{sim}{\sim} \psi$ if $\varphi \in GP_{n,n-1}F$ follows from [H5, Theorem 1.4]. By [H6, Proposition 6.16], there are examples of $\varphi \in GP_{n,l}F$, $1 \leq l \leq n - 2$, such that $\varphi \underset{stb}{\sim} \psi \not\Rightarrow \varphi \underset{hn}{\sim} \psi$.

(viii) The fact that in this situation generally $\varphi \underset{hn}{\sim} \psi \not\Rightarrow \varphi \underset{sim}{\sim} \psi$ has been shown in Sections 3 and 4. For counterexamples which show that generally $\varphi \underset{stb}{\sim} \psi \not\Rightarrow \varphi \underset{hn}{\sim} \psi$, cf. (vii).

Part (ix) follows from Corollary 2.9 and Theorem 3.4.

(x) If $\operatorname{ind} c(\varphi) = 1$ then $\varphi \in GP_3 F$, and if $\operatorname{ind} c(\varphi) = 2$ then $\varphi \in GP_{3,2} F$. In these cases, see (ix) and (vii). The remaining cases have been dealt with in Theorem 4.3 and Corollary 4.6. ∎

This list of known results can most definitely be extended even further, but we decided to stop at this point. One of the interesting open problems is whether $\varphi \underset{stb}{\sim} \psi$ always implies $\varphi \underset{bir}{\sim} \psi$ provided $\dim \varphi = \dim \psi$ (cf. [O], Sect. 3]). To our knowledge, there are no counterexamples known, and all the cases in which this implication has been established affirmatively are those found in the list above plus the case where φ and ψ are so-called special Pfister neighbors (see [AhO]).

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